



Vectors, velocity and displacement



Sample Modelling Activities with Excel and Modellus

ITforUS

(Information Technology for Understanding Science)

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I. Introduction

These set of activities illustrates how a modelling program like Modellus can be used to illustrate physical vector quantities and how these type of quantities relate to scalar quantities.

1. Background Theory

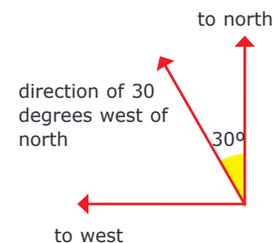
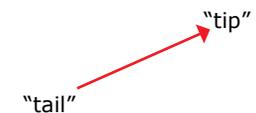
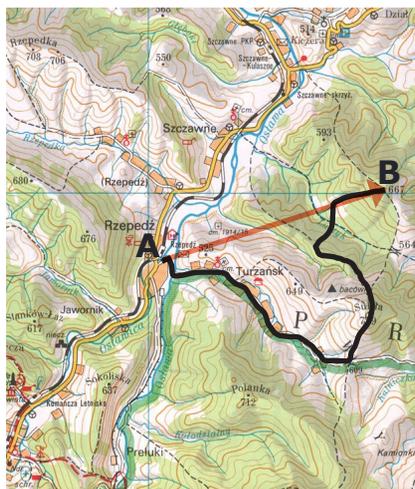
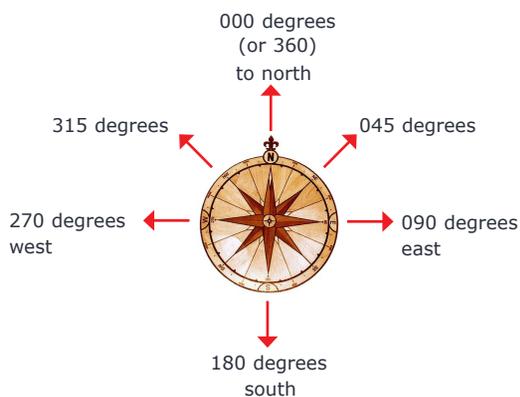
Physical quantities can be divided into two categories: **scalars** and **vectors**. While vector quantities to be fully described need both magnitude and direction, scalar quantities only need a real number.

For most common vector physical quantities, it is possible to describe magnitude and direction geometrically using **"arrows"**, a segment with a tail and a tip. The length of the arrow represents magnitude (it then needs a *scale!*) and the tip represents the direction. Displacement, velocity, acceleration, force are examples of vector quantities.

Distance is an example of a scalar quantity, in this case *bigger or equal to zero since it is a numerical description* of how far apart objects are at a certain moment in time. **Temperature** in the Kelvin scale is another example of a similar physical quantity.

The **direction** of a vector can be expressed as an angle of rotation of the vector about its tail. For example, once defined a certain direction, such as the northern direction, a vector can have a direction of 30 degrees west of north: that vector has been rotated 30 degrees towards the westerly direction.

The **most common convention** is to express the direction of a vector as a **counter clockwise angle of rotation about its "tail" from north due east**. Then, a vector with a direction of 90 degrees is a vector rotated 90 degrees in a counter clockwise direction from north relative to east. And a vector with a direction of 180 degrees has been rotated 180 degrees: it points south. Etc. In navigation, directions are usually expressed with three digits (from 000 to 360), as shown below.



The **distance between A and B** is a positive real number (a scalar): it measures the length of the straight line between A and B. The length of the trajectory (represented in black) is also a positive scalar. The **displacement** from A to B is a **vector**: its magnitude is equal to the distance between A and B and its direction says that the motion started in A and finished in B

A **vector can be multiplied by a scalar**. When such multiplication is done, the new vector has the same direction if the scalar is positive or the opposite direction if the scalar is negative.

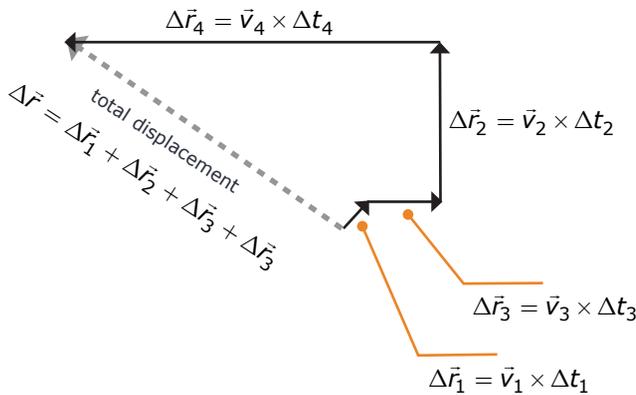
Multiplying velocity (a vector) by time interval (a positive scalar) gives displacement since velocity is the instantaneous rate of change of position (displacement). For constant velocity, we have

$$\vec{v} = \frac{\Delta \vec{r}}{\Delta t}$$

$$\Delta \vec{r} = \vec{v} \times \Delta t$$

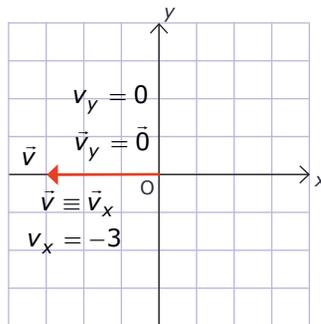
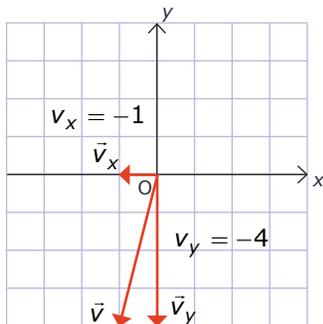
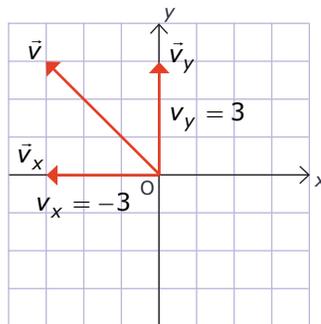
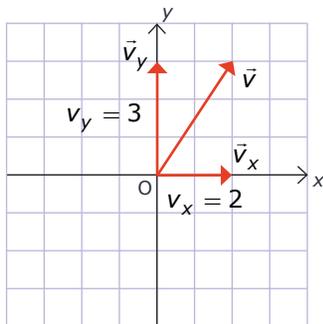
displacement = velocity \times time interval

If the velocity is not constant over a certain time interval, one needs to consider small time intervals where it can be considered constant (in reality or just as an approximation). In this conditions, the **total displacement** is the **sum of the displacements** in each small time interval:



Vector addition can be made placing the tail of one vector on the tip of the other: the vector sum connects the free tail to the free tip. This rule can be used successively, as shown on the picture.

In a plane, a **vector** can be represented by the **sum of two vector components**, one on the x axis and the other on the y axis. For each vector component, there is the correspondent **scalar component**, a real number. Four examples:



\vec{v} a vector...

v the magnitude of the vector...

$$v = \sqrt{v_x^2 + v_y^2}$$

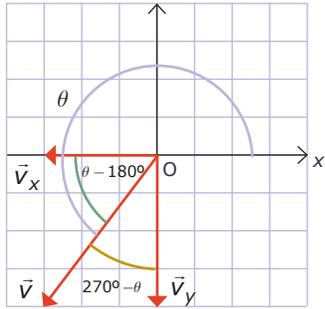
\vec{v}_x the vector component on the x axis

\vec{v}_y the vector component on the y axis

v_x the scalar component of the vector on the y axis (a real number)

v_y the scalar component of the vector on the x axis (a real number)

Scalar components of a vector can be expressed using trigonometric functions, as shown below:



$$v_x = v \times \cos(\theta)$$

$$v_y = v \times \sin(\theta)$$

$$v_x = v \times \cos(\theta - 180^\circ)$$

$$v_y = v \times \sin(\theta - 180^\circ)$$

$$v_x = v \times \sin(270^\circ - \theta)$$

$$v_y = v \times \cos(270^\circ - \theta)$$

This Module uses the idea of **numerical solution** of the equations of motion. Let's see how this idea works...

If we know the rate of change of quantity, which by definition is expressed by

$$\text{rate of change of a quantity} = \frac{\text{change in the quantity}}{\text{time taken to change}}$$

then we can write:

$$\text{change in the quantity} = \text{rate of change of the quantity} \times \text{time taken to change}$$

Since the change of a quantity can also be expressed as

$$\text{change in the quantity} = \text{new value of the quantity} - \text{previous value of the quantity}$$

we get, combining these two last equations:

$$\text{new value of the quantity} - \text{previous value of the quantity} = \text{rate of change of the quantity} \times \text{time taken to change}$$

$$\text{new value of the quantity} = \text{previous value of the quantity} + \text{rate of change of the quantity} \times \text{time taken to change}$$

Using symbols and considering velocity as the rate of change of position, the above reasoning can be represented as:

$$\begin{aligned} \vec{v} &= \frac{\Delta \vec{r}}{\Delta t} \\ \Delta \vec{r} &= \vec{v} \times \Delta t \\ \Delta \vec{r} &= \vec{r}_{t+\Delta t} - \vec{r}_t \\ \vec{r}_{t+\Delta t} - \vec{r}_t &= \vec{v} \times \Delta t \\ \vec{r}_{t+\Delta t} &= \vec{r}_t + \vec{v} \times \Delta t \end{aligned}$$

Using components, we have:

$$x_{t+\Delta t} = x_t + v_x \times \Delta t$$

$$y_{t+\Delta t} = y_t + v_y \times \Delta t$$

2. Science concepts introduced in this module

- Vector and scalar quantities
- Vector components
- Vector addition
- Position
- Velocity
- Trajectory
- Distance travelled
- Rates of change
- Iterative solution of the equations of motion

3. Other information

The activities concentrate on modelling motion without using a force law, i.e., an expression to compute the magnitude of the force. Learners only use velocity to compute next position. This is particular useful where the force law is not know or irrelevant, such as in navigation, in sea, land and air.

In the last activity learners are invited to make a model using acceleration. In this model, velocity in instant $t + \Delta t$ is computed from acceleration in instant t and time interval. And acceleration can be computed from force using Newton's fundamental law of motion.

Models with acceleration can be very useful to illustrate the mathematical meaning of inertia: acceleration change velocity... anf if acceleration if zero, velocity doesn't change...

II. Didactical approach

1. Pedagogical context

The activities presented in this module can be used with students of **upper secondary school, first year college students** and **secondary teachers**, either in Physics or in Mathematics classes.

They were not designed to fit in any curriculum. They simply illustrate how two interactive computer tools (Modellus and a spreadsheet like Excel) can be used to model physical phenomena. They can be **particularly useful for simultaneous training of Physics and Mathematics teachers**, promoting interdisciplinarity and reflection about concepts and representations and for the **introduction of simple numerical methods**.

2. Common student difficulties

Some of the student difficulties include:

- **Interpreting graphs** with time as the independent variable plotted on the horizontal axis.
- Using **rates of change** to define equivalent **iterative equations**.
- Computing **vector components**.

3. Evaluation of ICT

Computers are now the most common scientific tool, used in almost all aspects of the scientific endeavour, from measuring and modelling to writing and synchronous communication. It should then be *natural* to use computers in learning science.

Tools like Modellus or spreadsheets are particularly useful when large number of repetitive calculations need to be made, such as variables that are incrementally changed, and data must be presented graphically. In the case of Modellus, learners can also use vectors to represent rates of change and interactively see how these rates of change affect the values of physical quantities.

Using both Modellus and Excel allows the learner to compare how different programs implement the same mathematical idea. This helps learners to get familiar with *the* mathematical idea, not only the specific syntax of a particular program.

4. Teaching approaches

Good classroom organization is an essential component in a successful teaching approach, particularly when using complex tools such as computers and software. Most approaches to classroom organization that can give good results **mix features of students' autonomous work**, both individually and in small groups, to **teacher lecturing** to all class.

Typically, teachers can start with an all class approach, with students following the lesson with a screen projector. It is almost always a good idea to *ask one or more students to work on the computer connected to the projector*. This allows the teacher to have direct information of students' difficulties when manipulating the software and to be slower on the explanation of the ideas and activities that are being presented.

As all teachers know by experience, it is usually difficult for most students to follow written instructions, even when these instructions are only a few sentences long. To overcome this difficulty, teachers can ask students to **read the activities before** starting them and then promote a collective or group **discussion about what is supposed to be done with the computer**. *As a rule of thumb, students should only start an activity when they know what they will do on the activity: they will only consult the written worksheet just for checking details, not for following instructions.*

III. Activities

Where does it go?

The first activity is a **visual explanation** about the meaning of two vector quantities (**velocity** and **displacement**) and the relation between them.

It is important to get learners familiar with **nautical miles and knots**. If necessary, the tutor can mention that a nautical mile is simply a minute of arc (1/60 of 1/90 of the arc between the equator and the pole) along a meridian of the Earth.

The final point will be different... but the distance travelled will be the same! **Distance travelled and displacement are two different physical quantities.**

Direction can be represented in different ways. Traditionally, in navigation, people use angles measured from the top, increasing in the same direction as a hand clock. Learners can practise how familiar they are with the correspondence between the direction measured by the angle and the direction expressed using cardinal points.

Learners should be encouraged to use a few times the vector equation displacement = velocity × time interval and express the quantities as vectors in a correct scale.

On this second page learners can explore how to **compute successive displacements** with different velocities (different in magnitude and, or, different in direction).

Careful exploration of this page illustrates **how does an iterative process works**: the new value of position = old value of position + change, where change is displacement... and displacement is computed as the product of the rate of change of position (i.e., velocity) times time interval.

Learners can explore different vectors and different time intervals, building similar tables.

A1-1 OK I need to do this again... I need tutor support

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Where does it go? Displacement and velocity as vectors

- A boat is sailing on open sea...
- The speed is 10.0 knots (a knot is 1 nautical mile per hour, 1.852 km/h) and is heading 045 degrees (this means it is pointing northeast).
- Where is the boat after 10 hours sailing, if nothing changes?
- If the boat was heading in another direction, the displacement would be the same (100 miles), but the final point would be on any of the points on a circumference centered on the starting point...
- Velocity can't be treated as having only magnitude (a numerical value...), it also has **direction** (where does it point to?).
- The same is true for **displacement**: it has **magnitude** (how many miles from the starting point?) and **direction** (in what line from the starting point is the final point?).

magnitude of displacement = 10 miles/h × 10 h = 100 miles
 $\Delta r = v \times \Delta t$
 displacement = velocity × time interval
 distance travelled = 100 miles
 How big is this velocity, if the scale is the same of the above velocity?
 Where would the boat be after 5 hours sailing in this direction?

A1-1

A1-2 OK I need to do this again... I need tutor support

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Where does it go? Changing velocity...

- A boat is sailing on open sea...
- velocity: heading 045 degrees, speed 5.0 knots. Sailing this velocity for 1 hour...
- ... new velocity: heading 090 degrees, speed 5.0 knots. Sailing this velocity for 2 hours...
- ... new velocity: heading 135 degrees, speed 7.0 knots. Sailing this velocity for 2 hours...
- ... new velocity: heading 270 degrees, speed 10.0 knots. Sailing this velocity for 5 hours...
- How far from the starting point is the final point? Where is it? Why? What was the total distance travelled by the boat?

velocity	time of travel	displacement	distance travelled
5.0 knots 045°	1 h	5.0 miles 0.9 miles	5 miles (5 miles west-north-east...)
5.0 knots 090°	2 h	10.0 miles 0 miles	10 miles (10 miles west-north-east...)
7.0 knots 135°	1.5 h	10.5 miles 1.5 miles	21 miles (21 miles west-north-east...)
10.0 knots 270°	5 h	50 miles 0 miles	50 miles (50 miles west-north-east...)

86 miles (total)
 angle with north = 146° = 90° + 56°
 angle with north, measured from north = 360° - 56° = 304°

A1-2

A model of the motion of the boat

These two activities guide learners to build a simple **iterative model** in Modellus of a **boat moving with a constant speed** of 10 units in a direction that points nordest (if north points to the top of the page).

Look carefully at the text of the model and of the **initial conditions and the initial values** of the parameters. The Control window needs to be changed (change the Max value of steps as well as the Model Type to iterative).

Learners should be encouraged to analyse other values for the initial values and even where is the final position of the boat for these different values.

This activity makes the iterative model of the motion of the boat more interactive (some learners tend to confound *iterative* with *interactive*; assure yourself that this does not happens!).

The direction of the velocity is now changeable, using a bar that can change the angle from -360° to $+360^\circ$. Learners can try different trajectories, including circular trajectories (or almost circular...).

A possible **extension** of this activity is adding another bar for v , the magnitude of velocity, on the Animation Window. Using the two bars, one for the angle and the other for speed, learners can **change interactively both properties of velocity**, direction and magnitude.

It also possible to change the model to use a vector to interactively change both properties of velocity. The image on the right shows a simple model that allows the user to change the velocity with the mouse. Try it now... or wait for the last page of this module where this is illustrated in more detail.

A model of the motion of the boat

- Create the following model in Modellus (Don't forget to select Iterative Model and Max number of steps = 10 using the Options... button on the Control).
- Create a particle (the boat...) with coordinates x and y and a vector (the displacement of the boat...) with components x and y .
- Run the model and check how it works.
- Create a table and inspect all the values of the different variables.
- How far did the boat move? What are the position of the boat at the end of the departure coast?

After using the above model, add this new iteration:

$$x = \text{last } x + vx \times dt$$

$$y = \text{last } y + vy \times dt$$

$$t = \text{last } t + dt$$

Check its value on the table or place a digital meter on the Animation Window to see how it changes (the square root and the delta (change symbol) can be obtained clicking on the buttons $\sqrt{\quad}$ and Δ on the Model Window). The variable *distance travelled* (be sure not to make spelling mistakes!) accumulates the change in the magnitude of displacement in each iteration.

Moving the boat with constant speed and different velocity...

- Let's now create a model with a control bar that will be used to change the angle of the velocity.
- Reuse the previous model or create the following model in Modellus (Don't forget to select Iterative Model and Max number of steps = 500 using the Options... button on the Control).
- Resizes the particle (the boat...) with coordinates x and y , place a "speed indicator" ("horizontal bar" on the Animation Window, give adequate properties to this object).
- Run the model and change the value of the angle.
- You can also place digital meters to check the distance travelled by the boat and the time spent.

- Use the "speed indicator" to change the angle that velocity makes with the horizontal.
- Try to move the particle in order to return to the initial point.
- See how distance travelled changes...
- Is velocity changed or not?
- With constant speed, it is easy to compute distance travelled; we just need to multiply the speed (the magnitude of velocity) by the time interval. Check this relation.

Parameters		case 1
v_x		10.00
dt		0.10
v_y		10.00

Initial values		case 1
x		0.00
y		0.00
t		0.00

Iterative solution of the equations of motion

These three pages synthesize the meaning of an iterative solution of the equations of motion, explains how they can be implemented in a spreadsheet and how an iterative solution is related with functions.

Probably, it is wise not to build the last model (page 7) since one must be very careful in order to make no mistakes... It should be enough to analyse it in detail and **discuss how time is "delayed" in each function**. A somewhat challenging project could be *implementing that model on the same spreadsheet* of the iterative model (page 6)!

This first page synthesizes how the iterative solution works in a two dimensional space and how the **iteration** is expressed both in **vector notation** and using **scalar components** in two axes.

This detailed explanation of how to build the spreadsheet must be carefully analysed and followed. Most spreadsheet users don't know basic mathematical syntax on a spreadsheet, such as writing the number π , and how to use a factor to change from degrees to radians. Check if that is not the case of the learners using this activity...

This "graph" is NOT the graph of a function. It is a **trajectory** in the Oxy plane. And, since it represents point on space, it must have equal scales on both axes. Is necessary, make also the graph of the values of x and y as functions of t . Learners can be invited to sketch this two graphs using the values on the table and discuss how they are difficult to interpret, since the trajectory is not a straight line. It could also be interesting to make the graph of total distance travelled as a function of time (see below).

A.5 Iterative solution of the equations of motion

- When an object is moving with constant velocity, after a time interval of Δt we know that its displacement will be $v \cdot \Delta t$...
- Displacement can be computed from previous displacement plus displacement on current time interval...
- This is the iterative solution of the equation of motion: each new value of the positional vector is computed from the previous value plus change during the current interval.

For the displacement vector: $r_{i+1} = r_i + v_i \cdot \Delta t$

For each component: $x_{i+1} = x_i + v_{xi} \cdot \Delta t$
 $y_{i+1} = y_i + v_{yi} \cdot \Delta t$

Δt represents the displacement from the origin O (i.e., the positional vector on the Oxy reference frame).

A.6 Numerical/iterative solution of the equations of motion in a spreadsheet

TIPS: To apply a value to all cells in a column, click on the cell and drag the corner of the cell down as far as you want it.

TIPS: To copy a formula from one cell to another, click on the cell and drag the corner of the cell down as far as you want it.

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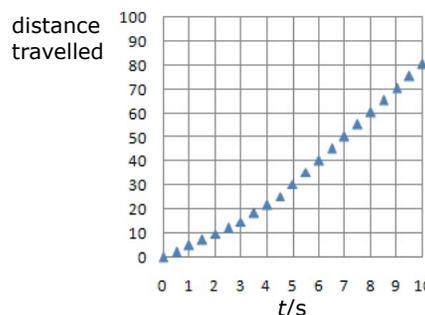
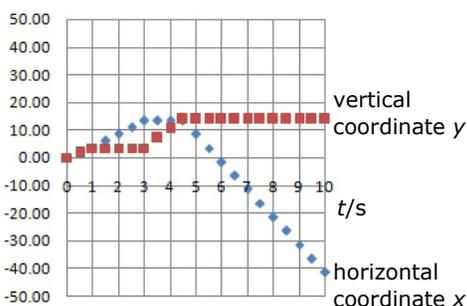
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A.7 Analytical solution (explicit function) of the equations of motion

- You don't need to write this model...
- An analytical solution is an explicit function of the independent variable (t , in the case of the motion of the boat)...
- For each sub-domain, it is necessary to define for the coordinates x and y a specific function of t , with the correct values of the parameters and angle for that sub-domain...
- The functions used in each sub-domain also need to take into consideration the initial values of x and y in each sub-domain in order to correctly describe the motion on the sub-domain.
- What type of models (explicit functions/analytical solutions or iterative models/numerical solutions) do you think must be used in computer games to move characters and objects? Give your reasons.



A simple and powerful model...

This final activity can be used to “play” with all the concepts discussed above.

Velocity is now represented as an **independent vector** that can be freely changed by the learner, both in magnitude and direction. Analyse how to get different trajectories, as the one shown (almost circular!).

An interesting discussion with this model can be about how these type of models can be used to *create computer games*.

It is important to note that **this model doesn't take inertia into account**. This means that velocity changes instantaneously, what is not true for real moving objects.

The model below presents a more correct model that takes shows how inertia can be modelled mathematically. In this more complete model, **each velocity component depends on the respective acceleration component**. The independent vector is now acceleration. Changing acceleration, both in magnitude and direction, moves the particle. Making the acceleration equal to 0, the particle continues moving, if it was moving... and remains stopped if it was stopped. Don't give big values to the magnitude of the acceleration... (is necessary, change the scale of the acceleration vector to a small scale, such as 1 pixel = 0.1 units).

